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Dependence of elastic constants of an anisotropic porous material upon porosity and fabric

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The elastic properties of an anisotropic porous material can be represented as functions of the material's solid volume fraction (or porosity) and the principal diameters of the material's fabric ellipsoid. The fabric ellipsoid is a measure of the anisotropy of the microstructure of a material. The definitions and measurement techniques for fabric ellipsoids in granular materials, foams, cancellous bone, and rocks are discussed. The principal results presented in this work are algebraic expressions for the dependence of the orthotropic elastic constants upon both solid volume fraction and the fabric ellipsoid.

1. Introduction

There are a number of natural and engineering materials which have a significant amount of porosity. Examples include granular materials (soil, sand, etc.), geological solids (rocks), sintered materials (ceramics, materials made by powder metallurgy), and cellular materials (wood, polymer foams, cancellous bone). For these materials, elastic constants have been found to have a strong dependence upon the solid volume fraction of the material. The solid volume fraction, V_v, is defined as

$$V_v = V_m/(V_m + V_p) \quad (1)$$

where V_m is the volume of the solid matrix, and V_p is the volume of the pores. Solid volume fraction is the additive reciprocal of porosity, p,

$$V_v = 1 - V_p/(V_m + V_p) = 1 - p \quad (2)$$

Many previous studies reported the relationship between elastic properties and solid volume fraction in different types of materials. These studies can be grouped into two classes: those which deal with low-porosity materials (porosity less than 50%) and those which deal with high-porosity materials (porosity greater than 50%). The low-porosity materials include porous sintered materials, rocks, ceramics and some granular materials. The high-porosity materials include polymer foams, cork, balsa wood, and cancellous bone.

Several studies have been carried out on the effect of variations of porosity on porous sintered materials. Spriggs [1] found empirically that the Young's modulus of porous sintered materials was exponentially dependent on porosity

$$E = E_0 \exp(-ap) \quad (3)$$

where E_0 is the zero-porosity Young's modulus and a is a constant. Wang [2] constructed a model of sintered materials which predicts the relationship between Young's modulus and porosity in these to be

$$E = E_0 \exp[-(a_0 p + a_1 p^2 + a_2 p^3 + \ldots)] \quad (4)$$

where a_0, a_1, a_2, etc. are material-dependent constants. Experimental evidence showed that terms in p of order higher than two could be neglected. Wang's [2] model also suggested that shear modulus, G, is proportional to Young's modulus or

$$G/G_0 = E/E_0 \quad (5)$$

where G_0 and E_0 are the zero-porosity shear and Young's moduli, respectively.

Most studies of high-porosity materials support a model which suggests that the Young's and shear moduli are proportional to the solid volume fraction raised to some power n

$$E = AV_v^n \quad (6)$$

$$G = BV_v^m \quad (7)$$

where A, B, n and m are material-dependent constants. Gibson and Ashby's [3] theoretical model of cellular materials showed that the Young's and shear moduli are dependent upon the square of solid volume fraction and, therefore, that shear modulus is proportional to Young's modulus. Their conclusion was based on the assumption that the predominant mode of deformation in foams is the bending of the structural elements. Their theoretical result is supported by experimental results from Baxter and Jones [4], Bensusan *et al.* [5], Brighton and Meazey [6], Chan and Nakamura [7], Gent and Thomas [8], Gibson [9], Moore *et al.* [10], and Phillips and Waterman [11]. Gibson and Ashby [3] also concluded that Poisson's ratio in cellular materials is independent of solid volume fraction. This conclusion is supported by experimental results from Gent and Thomas [8] and Gibson [9]. Patel [12] studied structural foams under the assumption that the predominant mode of defor-

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mation is axial deflection of the structural elements and his results suggest that Young's modulus is linearly dependent on solid volume fraction. The empirical study by Carter and Hayes [13] showed the Young's modulus to be dependent on the cube of the density in cancellous bone. This result is considered to be equivalent to a dependence upon the cube of solid volume fraction because the matrix material density of bone tissue is reasonably constant. However, Bensusan *et al.* [5] found the Young's modulus of cancellous bone to be proportional to the square of density and Williams and Lewis [14] found Young's modulus of cancellous bone to be linearly proportional to density.

The elastic properties of an elastic, anisotropic porous material are dependent both on its solid volume fraction and the geometrical organization of its structure. Therefore, if solid volume fraction is the first measure of local structure a second measure of the local structure is needed to characterize the geometrical anisotropy of the material. This topic is addressed in the following section.

2. Fabric

It is recognized that porosity or solid volume fraction is the primary measure of local material structure in a porous material. Porosity does not reflect any directionality of the specimen's structure. What then is the second best measure of local structure? This question was posed in the context of granular materials by Cowin [15]. Now there appears to be general agreement that a fabric ellipsoid is the best second measure of local material microstructure in many porous materials. In this paper the term fabric ellipsoid indicates any ellipsoid (i.e. any positive definite second-rank tensor in three dimensions) that characterizes the local anisotropy of the material's microstructure. The fabric ellipsoid is a point property (even though its measurement requires a finite test volume) and is therefore considered to be a continuous function of position in the material. Methods exist to measure fabric ellipsoids in cellular materials, rocks and granular materials and are described below.

In cellular materials, foams and cancellous bone, a fabric ellipsoid can be associated with the directional variation of a lineal measure called mean intercept length. The mean intercept length is the average distance between two solid/void interfaces in a given direction. A grid of parallel test lines is laid over a specimen in a given direction, as illustrated in Fig. 1, and the length of the test line is divided by the number of intercepts to give mean intercept length. Measurements are repeated for test lines orientated in several directions. The experimental procedure for this type of measurement is described by Whitehouse [16] and Harrigan and Mann [17]. Measurements are made on a plane of the material which is prepared to show contrast between the solid matrix and the pores. The methodology of making measurements is easily adapted to an automated computational system [17]. The fabric ellipsoid is constructed by measuring mean intercept length as a function of direction in three orthogonal planes. In each of the three planes the data are fitted to the equation of an ellipse. The three orthogonal ellipses that are formed are projections of an ellipsoid which is the fabric ellipsoid.

A different fabric measure is employed for granular materials. Oda [18], Oda *et al.* [19], and Satake [20] suggest that the best indicator of fabric in granular materials is the probability density function of the distribution of the orientation of contact normals, i.e. the normal at the point of contact between two granular particles, see Fig. 2. This distribution is found to be periodic with respect to direction and can be represented by an ellipsoid. The number of contact normals in a given direction is measured on an embedded section of a granular material. Data taken in a number of directions can be fitted to the equation of an ellipse. And the data from three orthogonal planes form the fabric ellipsoid.

An automated computational system has been developed to quantify fabric from electron micrographs of soil by Tovey [21]. This system examines the gradients of the intensity of an image at each spatial point. The distribution of gradient vectors with respect to direction plotted on a polar plot tends to correlate to an ellipse.

In the analysis of rock mechanics, the fabric is best determined by the orientation of cracks within the rock. A fabric ellipsoid can be formed from mean intercept length measurements taken from three orthogonal planes of a rock. This definition of fabric is similar to those proposed by Satake [20] and Harrigan

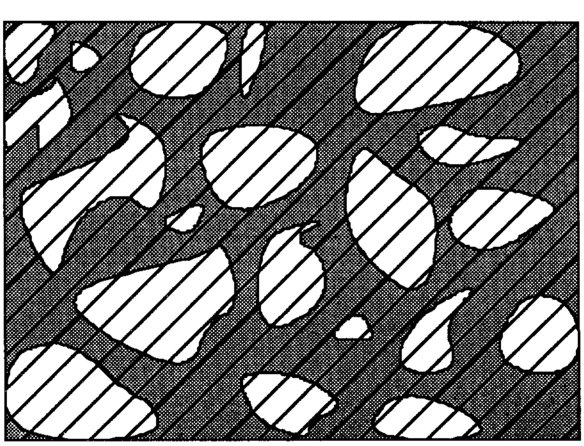

Figure 1 Test grid of parallel lines superimposed on a porous structure.

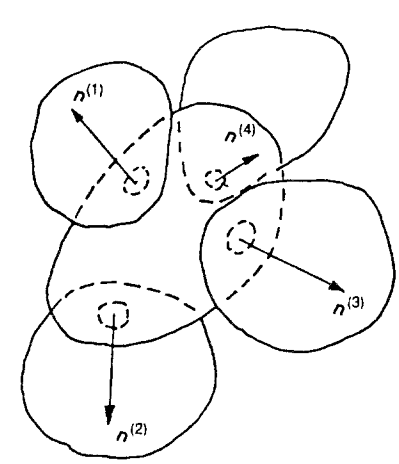

Figure 2 Contact normals, $n^{(i)}$, of a granular particle.

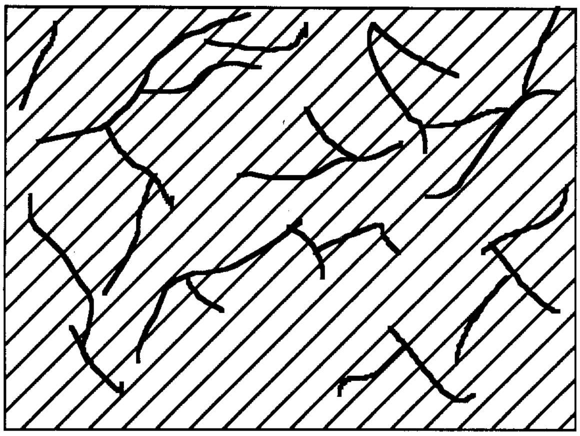

Figure 3 Test grid of parallel lines superimposed upon a rock.

and Mann [17]. Fig. 3 illustrates the measurement of mean intercept length in rock mechanics using a grid of parallel test lines superimposed on the surface of a rock. Oda [22] and Oda *et al.* [23] proposed a fabric ellipsoid for rock which incorporated the concentration of cracks as well as the directionality of cracks. In this definition the volume of the fabric ellipsoid is proportional to the concentration of cracks.

A fabric ellipsoid can be represented mathematically by a positive definite, symmetric second-rank tensor. The fabric tensor can be related to the fourth-rank elastic compliance tensor by tensor algebra. This relationship was derived by Cowin [24]. The results show that the orthotropic elastic moduli can be related to the lengths of the major axes of the fabric ellipsoid. The relationship between the fabric tensor and the compliance tensor is summarized in the following section and a derivation is given in Appendix 1.

3. Relationship between the fabric tensor and elastic properties

Let σ_1 to σ_6 and ε_1 to ε_6 represent the six components of stress σ_x, σ_y, σ_z, τ_{yz}, τ_{xz}, τ_{xy} and strain ε_x, ε_y, ε_z, $2\varepsilon_{yz}$, $2\varepsilon_{xz}$, $2\varepsilon_{xy}$, respectively. The orthotropic, linear elastic stress strain relation can then be written

$$\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \varepsilon_4\\ \varepsilon_5\\ \varepsilon_6\end{pmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & -\nu_{31}/E_3 & 0 & 0 & 0\\ -\nu_{12}/E_1 & 1/E_2 & -\nu_{32}/E_3 & 0 & 0 & 0\\ -\nu_{13}/E_1 & -\nu_{23}/E_2 & 1/E_3 & 0 & 0 & 0\\ 0 & 0 & 0 & 1/G_{23} & 0 & 0\\ 0 & 0 & 0 & 0 & 1/G_{31} & 0\\ 0 & 0 & 0 & 0 & 0 & 1/G_{12}\end{bmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{pmatrix} \quad (8)$$

where E_1, E_2, and E_3 are the Young's moduli, ν_{12}, ν_{21}, ν_{13}, ν_{31}, ν_{23}, and ν_{32} are the Poisson's ratios and G_{12}, G_{31}, and G_{23} are the shear moduli. The Young's moduli and Poisson's ratios satisfy the additional restrictions

$$\nu_{12}/E_1 = \nu_{21}/E_2, \quad \nu_{13}/E_1 = \nu_{31}/E_3, \quad \nu_{23}/E_2 = \nu_{32}/E_3 \quad (9)$$

Let A, B, and C denote the principal diameters of the fabric ellipsoid. Cowin [24] shows that if the matrix material of which the microstructure is composed is itself isotropic, then the principal diameters of the fabric ellipsoid must coincide with the normals to the three planes of reflective symmetry which characterize orthotropic material symmetry. The general relationship between the elastic constants and the fabric ellipsoid in the coordinate system which is composed of the normals to the three planes of reflective symmetry is found in Appendix 1 (Equation A4). Cowin [24] also shows that, if terms of order three and higher in A, B, and C are neglected, then the relationship between the orthotropic moduli E_1, E_2, E_3, G_1, G_2, G_3, ν_{12}, ν_{13}, ν_{23} and the solid volume fraction, V_v, and the fabric ellipsoid A, B, C can be expressed by

$$\begin{aligned}
1/E_1 &= g_1 + 2h_1 + (g_2 + 2h_2)I + (g_3 + 2h_3)I^2 \\
&\quad + (g_4 + 2h_4)II + 2(g_5 + 2h_5)A \\
&\quad + 2(g_6 + 2h_6)IA + (2g_7 + g_8 + 4h_7)A^2 \\
1/E_2 &= g_1 + 2h_1 + (g_2 + 2h_2)I + (g_3 + 2h_3)I^2 \\
&\quad + (g_4 + 2h_4)II + 2(g_5 + 2h_5)B \\
&\quad + 2(g_6 + 2h_6)IB + (2g_7 + g_8 + 4h_7)B^2 \\
1/E_3 &= g_1 + 2h_1 + (g_2 + 2h_2)I + (g_3 + 2h_3)I^2 \\
&\quad + (g_4 + 2h_4)II + 2(g_5 + 2h_5)C \\
&\quad + 2(g_6 + 2h_6)IC + (2g_7 + g_8 + 4h_7)C^2 \\
-\nu_{12}/E_1 &= g_1 + g_2 I + g_3 I^2 + g_4 II \\
&\quad + (g_5 + g_6 I)(A + B) \\
&\quad + g_7(A^2 + B^2) + g_8 AB \\
-\nu_{13}/E_1 &= g_1 + g_2 I + g_3 I^2 + g_4 II \\
&\quad + (g_5 + g_6 I)(A + C) \\
&\quad + g_7(A^2 + C^2) + g_8 AC \\
-\nu_{23}/E_2 &= g_1 + g_2 I + g_3 I^2 + g_4 II \\
&\quad + (g_5 + g_6 I)(B + C) \\
&\quad + g_7(B^2 + C^2) + g_8 BC \\
1/G_{23} &= h_1 + h_2 I + h_3 I^2 + h_4 II \\
&\quad + (h_5 + h_6 I)(B + C) + h_7(B^2 + C^2) \\
1/G_{31} &= h_1 + h_2 I + h_3 I^2 + h_4 II \\
&\quad + (h_5 + h_6 I)(C + A) + h_7(C^2 + A^2) \\
1/G_{12} &= h_1 + h_2 I + h_3 I^2 + h_4 II \\
&\quad + (h_5 + h_6 I)(A + B) + h_7(A^2 + B^2)
\end{aligned} \quad (10)$$

where the g and h are functions of V_v only and I, II, and III are the invariants of the fabric tensor

$$I = A + B + C, \quad II = AB + BC + AC, \quad III = ABC \quad (11)$$

4. Simplification of the relationship between elastic constants and fabric by assuming the Poisson's ratios to be independent of solid volume fraction

If Poisson's ratios are assumed to be independent of solid volume fraction, certain relationships between g and h in Equation 10 can be obtained. In porous materials this assumption is reasonable because the lateral contraction is due predominantly to mechanistic effects rather than a consequence of a volume preserving tendency. Physical arguments for this assumption in cellular materials and sintered materials are found in [2, 3, 8, 9]. Gibson and Ashby [3] developed a structural model to study the mechanics of rigid cellular materials. One of the conclusions reached through their analysis was that the Poisson's ratio of an isotropic cellular material was independent of the material's porosity. Experimental data from Gibson [9] and Gent and Thomas [8] show, statistically, no dependence of Poisson's ratio on porosity in isotropic cellular plastics. Wang [2] presented a theoretical analysis of sintered isotropic porous materials which showed Young's modulus, E, to be proportional to shear modulus, G, at all porosities. Poisson's ratio, ν, is related to E and G in an isotropic material by $\nu = E/2G - 1$ and if E and G are proportional, then ν is a constant.

We consider ν_{12} as a typical orthotropic Poisson's ratio. An expression for ν_{12} can be obtained by substitution for $1/E_1$ and $-\nu_{12}/E_1$ from Equation 10 into,

$$\nu_{12} = \left(\frac{\nu_{12}}{E_1}\right) E_1 \qquad (12)$$

thus

$$\nu_{12} = \frac{-[g_1 + g_2 I + g_3 I^2 + g_4 II + (g_5 + g_6 I)(A + B) + g_7(A^2 + B^2) + g_8 AB]}{[g_1 + 2h_1 + (g_2 + 2h_2)I + (g_3 + 2h_3)I^2 + (g_4 + 2h_4)II + 2(g_5 + 2h_5)A + 2(g_6 + 2h_6)IA + (2g_7 + g_8 + 4h_7)A^2]} \qquad (13)$$

The result of the division of these polynomials is

$$\nu_{12} = a + bI + cA + dB + eI^2 + fII + gA^2 + hB^2 + iIA + jIB + kAB + 0(3) + \ldots \qquad (14)$$

where

$$a = -g_1/(g_1 + 2h_1)$$
$$b = -2(g_2 h_1 - g_1 h_2)/(g_1 + 2h_1)^2$$
$$c = (g_1 g_5 - 2h_1 g_5 + 4g_1 h_5)/(g_1 + 2h_1)^2$$
$$d = -g_5/(g_1 + 2h_1)$$
$$e = -2[(g_3 h_1 - g_1 h_3)/(g_1 + 2h_1)^2 - (g_2 + 2h_2)(g_2 h_1 - g_1 h_2)/(g_1 + 2h_1)^3]$$
$$f = -2(g_4 h_1 - g_1 h_4)/(g_1 + 2h_1)^2$$
$$g = -[(-g_1 g_7 - g_1 g_8 - 4g_1 h_7 + 2h_1 g_7)/(g_1 + 2h_1)^2 - (-g_1 g_5 + 2h_1 g_5 - 4g_1 h_5) \times (2g_5 + 4h_5)/(g_1 + 2h_1)^3]$$
$$h = -g_7/(g_1 + 2h_1)$$
$$i = -[(-g_1 g_6 + 2g_6 h_1 - 4g_1 h_6)/(g_1 + 2h_1)^2 - 4(g_5 + 2h_5)(g_2 h_1 - g_1 h_2)/(g_1 + 2h_1)^3 - (g_2 + 2h_2)(-g_1 g_5 + 2h_1 g_5 - 4g_1 h_5)/(g_1 + 2h_1)^3]$$
$$j = -[g_6(g_1 + 2h_1) - g_5(g_2 + 2h_2)]/(g_1 + 2h_1)^2$$
$$k = -[g_8(g_1 + 2h_1) - 2g_5(g_5 + 2h_5)]/(g_1 + 2h_1)^2$$

If ν_{12} is assumed to be independent of solid volume fraction, then $a, b, c, d, e, f, g, h, i, j$, and k must be constants. By applying this argument it can be shown that $g_1, g_5, g_7, g_8, h_1, h_5$, and h_7 must be proportional. Thus

$$g_1 = f_1(V_v), \quad g_5 = c_1 f_1(V_v), \quad g_7 = c_2 f_1(V_v), \quad g_8 = c_3 f_1(V_v)$$
$$h_1 = c_4 f_1(V_v), \quad h_5 = c_5 f_1(V_v), \quad h_7 = c_6 f_1(V_v)$$

and

$$g_2 = f_2(V_v), \quad g_3 = f_3(V_v), \quad g_4 = f_4(V_v), \quad g_6 = f_5(V_v)$$
$$h_2 = f_6(V_v), \quad h_3 = f_7(V_v), \quad h_4 = f_8(V_v), \quad h_6 = f_9(V_v) \qquad (15)$$

The application of the assumption of constant Poisson's ratios thus reduces the number of material-dependent functions of solid volume fraction from 15 to 9 functions and 6 constants which must be determined experimentally. The resulting simplified relationship between the elastic constants and the fabric ellipsoid and the solid volume fraction can then be written as

$$1/E_1 = (1 + 2c_4)f_1 + (f_2 + 2f_6)I + (f_3 + 2f_7)I^2 + (f_4 + 2f_8)II + 2(c_1 + 2c_5)f_1 A + 2(f_5 + 2f_9)IA + (2c_2 + c_3 + 4c_6)f_1 A^2,$$
$$1/E_2 = (1 + 2c_4)f_1 + (f_2 + 2f_6)I + (f_3 + 2f_7)I^2 + (f_4 + 2f_8)II + 2(c_1 + 2c_5)f_1 B + 2(f_5 + 2f_9)IB + (2c_2 + c_3 + 4c_6)f_1 B^2,$$
$$1/E_3 = (1 + 2c_4)f_1 + (f_2 + 2f_6)I + (f_3 + 2f_7)I^2 + (f_4 + 2f_8)II + 2(c_1 + 2c_5)f_1 C + 2(f_5 + 2f_9)IC + (2c_2 + c_3 + 4c_6)f_1 C^2,$$
$$-\nu_{12}/E_1 = f_1 + f_2 I + f_3 I^2 + f_4 II + (c_1 f_1 + f_5 I) \times (A + B) + c_2 f_1(A^2 + B^2) + c_3 f_1 AB,$$
$$-\nu_{13}/E_1 = f_1 + f_2 I + f_3 I^2 + f_4 II + (c_1 f_1 + f_5 I)$$

$$\times (A + C) + c_2 f_1 (A^2 + C^2) + c_3 f_1 AC,$$
$$-\nu_{23}/E_2 = f_1 + f_2 I + f_3 I^2 + f_4 II + (c_1 f_1 + f_5 I) \times (B + C) + c_2 f_1 (B^2 + C^2) + c_3 f_1 BC,$$
$$1/G_{23} = c_4 f_1 + f_6 I + f_7 I^2 + f_8 II + (c_5 f_1 + f_9 I) \times (B + C) + c_6 f_1 (B^2 + C^2),$$
$$1/G_{31} = c_4 f_1 + f_6 I + f_7 I^2 + f_8 II + (c_5 f_1 + f_9 I) \times (C + A) + c_6 f_1 (C^2 + A^2),$$
$$1/G_{12} = c_4 f_1 + f_6 I + f_7 I^2 + f_8 II + (c_5 f_1 + f_9 I) \times (A + B) + c_6 f_1 (A^2 + B^2) \quad (16)$$

5. Simplification of the relationship between elastic constants and fabric by normalizing the fabric tensor

In some cases it is desirable to normalize the fabric tensor. From his model, Patel [12] concluded that the mechanical properties of rigid foams were independent of pore size. Similarly, in their theoretical model for cellular plastics, Gibson and Ashby [3] assumed that the elastic properties are independent of the absolute dimensions of the microstructure. Also, from his model for sintered porous materials, Wang [2] showed that elastic properties were independent of particle size. All of the above studies suggest that normalization of the fabric tensor is valid in many porous materials. The fabric tensor is normalized by the requirement

$$I = A + B + C = 1. \quad (17)$$

By applying the requirement 17 to Equation 10, Equation 10 becomes

$$1/E_1 = k_1 + 2k_6 + (k_2 + 2k_7)II + 2(k_3 + 2k_8)A + (2k_4 + k_5 + 4k_9)A^2$$
$$1/E_2 = k_1 + 2k_6 + (k_2 + 2k_7)II + 2(k_3 + 2k_8)B + (2k_4 + k_5 + 4k_9)B^2$$
$$1/E_3 = k_1 + 2k_6 + (k_2 + 2k_7)II + 2(k_3 + 2k_8)C + (2k_4 + k_5 + 4k_9)C^2$$
$$-\nu_{12}/E_1 = k_1 + k_2 II + k_3(A + B) + k_4(A^2 + B^2) + k_5 AB$$
$$-\nu_{13}/E_1 = k_1 + k_2 II + k_3(A + C) + k_4(A^2 + C^2) + k_5 AC$$
$$-\nu_{23}/E_2 = k_1 + k_2 II + k_3(B + C) + k_4(B^2 + C^2) + k_5 BC$$
$$1/G_{23} = k_6 + k_7 II + k_8(B + C) + k_9(B^2 + C^2)$$
$$1/G_{31} = k_6 + k_7 II + k_8(C + A) + k_9(C^2 + A^2)$$
$$1/G_{12} = k_6 + k_7 II + k_8(A + B) + k_9(A^2 + B^2) \quad (18)$$

where $k_1 = g_1 + g_2 + g_3$, $k_2 = g_4$, $k_3 = g_5 + g_6$, $k_4 = g_7$, $k_5 = g_8$, $k_6 = h_1 + h_2 + h_3$, $k_7 = h_4$, $k_8 = h_5 + h_6$, and $k_9 = h_7$.

A system of nine equations is produced with nine functions of solid volume fraction. These nine equations are not linearly independent. At a given density, seven unique solutions can be solved for directly. From these solutions five of the nine ks (k_3, k_4, k_5, k_8, and k_9) can be derived. The remaining ks (k_1, k_2, k_6, and k_7) are solved for using linear regression on data from a number of specimens. A method for solving the Equations 18 is given in Appendix 2.

If it is also assumed the Poisson's ratios are independent of solid volume fraction the nine ks are

$$k_1 = f_1(V_v), \quad k_3 = c_1 f_1(V_v), \quad k_4 = c_2 f_1(V_v),$$
$$k_5 = c_3 f_1(V_v), \quad k_6 = c_4 f_1(V_v), \quad k_8 = c_5 f_1(V_v),$$
$$k_9 = c_6 f_1(V_v), \quad k_2 = f_2(V_v), \quad k_7 = f_3(V_v) \quad (19)$$

This demonstrates the further reduction to three functions of V_v and six material dependent constants which must be determined experimentally. The relationship between the orthotropic elastic constants and the fabric ellipsoid and solid volume fraction as a result of the simplifying assumptions of constant Poisson's ratios and normalized fabric tensor is then

$$1/E_1 = (1 + 2c_4)f_1 + (f_2 + 2f_3)II + 2(c_1 + 2c_5)f_1 A + (2c_2 + c_3 + 4c_6)f_1 A^2$$
$$1/E_2 = (1 + 2c_4)f_1 + (f_2 + 2f_3)II + 2(c_1 + 2c_5)f_1 B + (2c_2 + c_3 + 4c_6)f_1 B^2$$
$$1/E_3 = (1 + 2c_4)f_1 + (f_2 + 2f_3)II + 2(c_1 + 2c_5)f_1 C + (2c_2 + c_3 + 4c_6)f_1 C^2$$
$$-\nu_{12}/E_1 = f_1 + f_2 II + c_1 f_1(A + B) + c_2 f_1(A^2 + B^2) + c_3 f_1 AB$$
$$-\nu_{13}/E_3 = f_1 + f_2 II + c_1 f_1(A + C) + c_2 f_1(A^2 + C^2) + c_3 f_1 AC$$
$$-\nu_{23}/E_2 = f_1 + f_2 II + c_1 f_1(B + C) + c_2 f_1(B^2 + C^2) + c_3 f_1 BC$$
$$1/G_{23} = c_4 f_1 + f_3 II + c_5 f_1(B + C) + c_6 f_1(B^2 + C^2)$$
$$1/G_{31} = c_4 f_1 + f_3 II + c_5 f_1(C + A) + c_6 f_1(C^2 + A^2)$$
$$1/G_{12} = c_4 f_1 + f_3 II + c_5 f_1(A + B) + c_6 f_1(A^2 + B^2) \quad (20)$$

This model facilitates relatively easy experimental quantification of the relationship between elastic properties and microstructure for a given material.

6. Conclusions

A general relationship between the elastic constants and the solid volume fraction and fabric of an orthotropic porous solid was reported by Cowin [24]. The fabric can be represented by a fabric ellipsoid which represents the geometric anisotropy of the porous structure. Methods to measure fabric ellipsoids in cellular materials, foams, cancellous bone, granular materials, soils and rocks have been discussed.

The general relationship is simplified by:

1. assuming that the Poisson's ratios are independ-

ent of solid volume fraction. Studies which support this assumption in foams and sintered materials were discussed;

2. assuming that the fabric ellipsoid could be normalized.

If both of these assumptions are valid in a given material the experimental determination of three functions of solid volume fraction and six material-dependent constants is required to characterize the general relationship. To determine these constants the elastic properties, the fabric ellipsoid, and the solid volume fraction must be determined for a number of specimens at various porosities.

Appendix 1

A relationship between the orthotropic elastic constants and the fabric ellipsoid was derived by Cowin [24]. The fabric ellipsoid can be represented by a positive definite, symmetric second rank tensor. This fabric tensor is denoted $\boldsymbol{H}$. The development of this relationship is based on the assumption that the solid material which makes up the porous material's microstructure is isotropic and that the anisotropy of the material itself is due only to the geometry of the microstructure represented by the fabric tensor. The mathematical statement of this notion is that the stress tensor, $\boldsymbol{\sigma}$, is an isotropic function of the strain tensor, $\boldsymbol{\varepsilon}$, and the fabric tensor, $\boldsymbol{H}$, as well as the solid volume fraction, V_v. Thus the tensor valued function

$$\boldsymbol{\sigma} = \sigma(V_v, \boldsymbol{\varepsilon}, \boldsymbol{H}) \qquad \text{(A1)}$$

has the property that

$$\boldsymbol{Q\sigma Q}^{\mathrm{T}} = \sigma(V_v, \boldsymbol{Q\varepsilon Q}^{\mathrm{T}}, \boldsymbol{QHQ}^{\mathrm{T}}) \qquad \text{(A2)}$$

for the orthogonal tensors $\boldsymbol{Q}$. This definition of an isotropic tensor valued function is that given, for example, by Truesdell and Noll [25].

The most general form of the relationship between the compliance tensor and the fabric tensor consistent with the isotropy assumption described above is

$$\begin{aligned} K_{ijkm} = {} & a_1\delta_{ij}\delta_{km} + a_2(H_{ij}\delta_{km} + \delta_{ij}H_{km}) \\ & + a_3(\delta_{ij}H_{kq}H_{qm} + \delta_{km}H_{iq}H_{qj}) + b_1H_{ij}H_{km} \\ & + b_2(H_{ij}H_{kq}H_{qm} + H_{is}H_{sj}H_{km}) \\ & + b_3H_{is}H_{sj}H_{kq}H_{qm} + c_1(\delta_{ki}\delta_{mj} + \delta_{mi}\delta_{kj}) \\ & + c_2(H_{ik}\delta_{mj} + H_{kj}\delta_{mi} + H_{im}\delta_{kj} + H_{mj}\delta_{ki}) \\ & + c_3(H_{ir}H_{rk}\delta_{mj} + H_{kr}H_{rj}\delta_{mi} \\ & + H_{ir}H_{rm}\delta_{kj} + H_{mr}H_{rj}\delta_{ik}), \qquad \text{(A3)} \end{aligned}$$

where K_{ijkm} is the fourth rank compliance tensor in indicial notation and a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, and c_3 are functions of V_v and tr$\boldsymbol{H}$, tr$\boldsymbol{H}^2$ and tr$\boldsymbol{H}^3$.

The representation (Equation A3) for the fourth-rank compliance tensor is not capable of representing all possible elastic material symmetries. It cannot represent triclinic material symmetry, which is total lack of symmetry, or monoclinic symmetry which is characterized by a single plane of reflective material symmetry. The least material symmetry that can be represented by Equation A3 is orthotropy. To see that this is the case we expand Equation A3 in the indicial notation in the coordinate system that diagonalizes the fabric tensor. Thus the H_{12}, H_{23}, and H_{13} components of $\boldsymbol{H}$ vanish in this system and H_{11}, H_{22}, and H_{33} are the three eigenvalues of $\boldsymbol{H}$. The result of this expansion of Equation A1 is the following nine non-zero components

$$\begin{aligned} K_{1111} &= a_1 + 2c_1 + 2(a_2 + 2c_2)H_{11} \\ &\quad + (2a_3 + b_1 + 4c_3)H_{11}^2 + 2b_2H_{11}^3 + b_3H_{11}^4 \\ K_{2222} &= a_1 + 2c_1 + 2(a_2 + 2c_2)H_{22} \\ &\quad + (2a_3 + b_1 + 4c_3)H_{22}^2 + 2b_2H_{22}^3 + b_3H_{22}^4 \\ K_{3333} &= a_1 + 2c_1 + 2(a_2 + 2c_2)H_{33} \\ &\quad + (2a_3 + b_1 + 4c_3)H_{33}^2 + 2b_2H_{33}^3 + b_3H_{33}^4 \\ K_{1122} &= a_1 + a_2(H_{11} + H_{22}) + a_3(H_{11}^2 + H_{22}^2) \\ &\quad + b_1H_{11}H_{22} + b_2(H_{11}H_{22}^2 + H_{22}H_{11}^2) \\ &\quad + b_3H_{11}^2H_{22}^2 \\ K_{1133} &= a_1 + a_2(H_{11} + H_{33}) + a_3(H_{11}^2 + H_{33}^2) \\ &\quad + b_1H_{11}H_{33} + b_2(H_{11}H_{33}^2 + H_{33}H_{11}^2) \\ &\quad + b_3H_{11}^2H_{33}^2 \\ K_{2233} &= a_1 + a_2(H_{22} + H_{33}) + a_3(H_{22}^2 + H_{33}^2) \\ &\quad + b_1H_{22}H_{33} + b_2(H_{22}H_{33}^2 + H_{33}H_{22}^2) \\ &\quad + b_3H_{22}^2H_{33}^2 \\ K_{1212} &= c_1 + c_2(H_{11} + H_{22}) + c_3(H_{11}^2 + H_{22}^2) \\ K_{1313} &= c_1 + c_2(H_{11} + H_{33}) + c_3(H_{11}^2 + H_{33}^2) \\ K_{2323} &= c_1 + c_2(H_{22} + H_{33}) + c_3(H_{22}^2 + H_{33}^2) \end{aligned} \qquad \text{(A4)}$$

and that all other components K_{ijkm} vanish. If K_{ijkm} is expanded to a coordinate system that does not diagonalize the fabric tensor, 21 non-zero components exist. Therefore, it is imperative experimentally that elastic properties be measured in a coordinate system which diagonalizes $\boldsymbol{H}$.

In the exact representation for K_{ijkm} above there exist terms in $\boldsymbol{H}$ up to order four. However, it is shown by Cowin [24] that this representation can be approximated by retaining terms up to order two. In this approximation a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, and c_3 are given by

$$\begin{aligned} a_1(V_v, I, II, III) &= g_1(V_v) + g_2(V_v)I + g_3(V_v)I^2 \\ &\quad + g_4(V_v)II \\ a_2(V_v, I, II, III) &= g_5(V_v) + g_6(V_v)I \\ a_3(V_v, I, II, III) &= g_7(V_v) \\ b_1(V_v, I, II, III) &= g_8(V_v) \\ b_2(V_v, I, II, III) &= 0 \\ b_3(V_v, I, II, III) &= 0 \\ c_1(V_v, I, II, III) &= h_1(V_v) + h_2(V_v)I + h_3(V_v)I^2 \\ &\quad + h_4(V_v)II \\ c_2(V_v, I, II, III) &= h_5(V_v) + h_6(V_v)I \\ c_3(V_v, I, II, III) &= h_7(V_v) \end{aligned} \qquad \text{(A5)}$$

where g and h are functions of V_v only, and where I, II, and III are the invariants of $\boldsymbol{H}$ related to tr$\boldsymbol{H}$, tr$\boldsymbol{H}^2$ and tr$\boldsymbol{H}^3$ by

$$I = \operatorname{tr}\boldsymbol{H},\ II = 1/2((\operatorname{tr}\boldsymbol{H})^2 - \operatorname{tr}\boldsymbol{H}^2)$$
$$III = 1/6((\operatorname{tr}\boldsymbol{H})^3 - 3\operatorname{tr}\boldsymbol{H}\operatorname{tr}\boldsymbol{H}^2 + 2\operatorname{tr}\boldsymbol{H}^3) \quad \text{(A6)}$$

Appendix 2

At a given solid volume fraction the values of k_3, k_4, k_5, k_8, and k_9 can be determined directly by breaking the nine equations in Equation 19 into three sets of linearly independent equations and solving those equations. The values of k_8 and k_9 are found by solving

$$1/G_{12} = \alpha_7 + \alpha_8(A + B) + \alpha_9(A^2 + B^2)$$
$$1/G_{13} = \alpha_7 + \alpha_8(A + C) + \alpha_9(A^2 + C^2)$$
$$1/G_{23} = \alpha_7 + \alpha_8(B + C) + \alpha_9(B^2 + C^2) \quad \text{(A6)}$$

where $\alpha_7 = k_6 + k_7 II$, $\alpha_8 = k_8$, and $\alpha_9 = k_9$. The value of k_3 can be found by solving

$$1/E_1 = \alpha_1 + \alpha_2 A + \alpha_3 A^2$$
$$1/E_2 = \alpha_1 + \alpha_2 B + \alpha_3 B^2$$
$$1/E_3 = \alpha_1 + \alpha_2 C + \alpha_3 C^2 \quad \text{(A7)}$$

where $\alpha_1 = k_1 + 2k_6 + (k_2 + 2k_7)II$, $\alpha_2 = 2(k_3 + 2k_8)$, and $\alpha_3 = (2k_4 + k_5 + 4k_9)$. Therefore,

$$k_3 = \alpha_2/2 - 2k_8 \quad \text{(A8)}$$

The values of k_4 and k_5 are found by solving

$$-\nu_{12}/E_1 - k_3(A + B) = \alpha_4 + \alpha_5(A^2 + B^2) + \alpha_6 AB$$
$$-\nu_{13}/E_1 - k_3(A + C) = \alpha_4 + \alpha_5(A^2 + C^2) + \alpha_6 AC$$
$$-\nu_{23}/E_2 - k_3(B + C) = \alpha_4 + \alpha_5(B^2 + C^2) + \alpha_6 BC \quad \text{(A9)}$$

where $\alpha_4 = k_1 + k_2 II$, $\alpha_5 = k_4$, and $\alpha_6 = k_5$.

The values of k_1, k_2, k_6, and k_7 can be found from α_4 and α_7 using multiple regression methods on data from many specimens.

$$\alpha_4(V_v) = k_1(V_v) + k_2(V_v)II, \text{ and}$$
$$\alpha_7(V_v) = k_6(V_v) + k_7(V_v)II \quad \text{(A10)}$$

A double regression, with V_v and II as independent variables, applied to the above equations can be used to determine k_1, k_2, k_6, and k_7.

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